Rethinking Calculus: Learning and Thinking

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1. Remodeling Calculus The Institution. Surely the renewal of Calculus is a good idea, one good enough to attract the attention and energy of many good people. But this is *Calculus the Institution* - that peculiarly American academic event and all its supporting structures and expectations. Professor Knisely, however, barely hints at matters of institutional implementation, so I conclude that he is addressing *Calculus, the System of Knowledge and Technique.* As such, his paper is, perhaps, a warm-up exercise to a deep and long overdue reconsideration of the appropriate intellectual content of Calculus, one that has been postponed while we attempt to remodel Calculus the Institution.

This remodeling has proven to be an arduous task for two reasons: (1) the renovation is taking place whilst the owners and stakeholders continue to inhabit the institution (a constraint applying to most educational reform); and relatedly (2) we have left all the larger structural features of the institution intact, including those features that connect it to the outside world, e.g., to the rapidly changing K-12 education. *The basic architecture and its place in the larger world are untouched.* I suggest that we embark on the more fundamental rebuilding towards which Knisely points. In so doing we need to come to terms with the *relations*, existing and possible, between Calculus the Institution (C-INST) and Calculus the System of Knowledge and Technique (C-KNOWL). And we need to look more deeply and critically at the assumptions, largely tacit, that hold the status quo in place and provide some concrete, implementable alternatives.

2. Relations Between C-KNOWL and C-INST. The key relation of interest to me involves learning and cognition. How can ideas and techniques become knowable and usable by those who need to know and use them? And who needs to know them, and in what ways do they need to know them? But before we can get to these questions, we must review the other key relation between C-KNOWL and C-INST, the historical one.

C-INST is the product of several centuries of evolution. The curriculum and texts are rooted in C-KNOWL, which developed at the hands of masters in the 17th and 18th centuries. Many basic curricular structures set down in textbooks by L'Hopital, the Bernoulli's, Euler, and their contemporaries, have remained largely invariant through the 20th century - for very good reason: they served traditional purposes and populations extremely well. Indeed, this presentation of C-KNOWL is at the foundation of our civilization's scientific and technological infrastructure. While C-KNOWL evolved into an almost sacred academic tradition [**7**], the ambient societies, the nature of education, and the relations between education and the larger society, including and especially in the United States, changed and continue to change profoundly. It is worth noting that, according to Department of

Education statistics [**13**], the percentage of students taking AP Calculus today is equal to the percentage graduating from high school a century ago! And, as recently as the 1950's, immediately prior to the huge increases in US access to higher education, calculus was commonly preceded by a preparatory course even at elite universities. Our expectations regarding who can learn what surely change with the times. The C-INST we know today, while connected to a venerable C-KNOWL, is a relatively recent artifact. Its increasing dysfunction and ill-fit with the new circumstances, especially technological ones, are what gave rise to the Calculus reform movement - the remodeling of C-INST.

3. Calculus Reform for the Other 90%. The fixing of C-INST serves only 10% or so of our population, the socio-economic and intellectual elite. The population at large - 4 million in each age-level cohort - continues to be denied access to the key ideas of C-KNOWL, a quietly accepted national oversight, despite the fact that we collectively spend billions of dollars supporting a curriculum largely aimed at calculus [**9**]. At the same time we congratulate ourselves on increasing percentages of students passing AP Calculus - to perhaps 3-4% of a given grade-level cohort [**13**]. Not only do we ignore 90% of the population's calculus learning, the 10% on whom we do focus actually need to learn much more mathematics of change and variation than C-INST currently allows them to encounter, for example, the mathematics of dynamical systems: the ideas, representations, and skills needed to make sense of nonlinear phenomena - the mathematics that flourishes in the computational medium [**1**, **11**]. And perhaps even some of the ideas Knisely suggests.

This is the background against which I wish to discuss learning and thinking of the mathematics of change and variation, including calculus. It is not enough to toss around names of ideas, procedures, and relationships that exist in the formal cultural record of mathematical achievement and in some form or other in the minds of that super-elite constituting professional mathematicians. As the late Morris Kline reminded us, the post-hoc logical structures of these mathematical products may have little to do with the structure of the experiences that students need in order to build viable versions of them in their minds [**5**]. The kind of idle speculation or assertion about the former, without regard for the latter, that appears in Knisely's article represents a general and entirely understandable tendency in our community to think and plan in terms of the cultural artifacts and language that we inhabit. The alert reader will have noticed that I indulged in a bit of this in the previous paragraph when mentioning dynamical systems. As epistemological Flatlanders, we don't distinguish between "up" and "North." But at some point, preferably earlier rather than later, our analyses must turn to learning and thinking, to the conceptual, cultural, and experiential roots of our mathematics - we must break our mind-forged manacles and look *up*.

Each word or phrase that we use to denote some mathematics is but a pointer to a thick web of ideas and relationships, treasures hard-won by great mathematical minds and requiring even greater struggles to be understood by more ordinary minds. I suggest that when we take learning and thinking seriously, we quickly dig to the foundations of our discipline. Simultaneously, we begin the rethinking that reaches beyond remodeling C-INST to build the extended means by which the neglected and under-served 90% may come to know some of the classical C-KNOWL, by which our favored (and important) 10% may come to know it more deeply, and by which both groups may come to learn an even broader and richer mathematics of change and variation. Such rethinking requires an openness to new organizations of ideas, new notations and ways of acting on notations, new uses of interactive technologies that reach beyond the CAS's designed to facilitate or supplant traditional adult competencies with formal symbols, new time scales for learning big ideas (years instead of months), and an enriched conception of what counts as legitimate mathematical thinking. Our current work is beginning to shed light on the transformative power of such ideas as change and variation to contextualize and organize many of the ideas and skills in K-12 mathematics already regarded as important, and to reveal the efficiencies in curricular organization that are required in order to make room for the new mathematics needed by students living their lives into the second half of the 21st century.

4. Rethinking Calculus - An Illustration.[1] To illustrate I sketch briefly some approaches developed in the ongoing SimCalc Project, with no pretense that it is complete or definitive. We began with a combination of historical analysis that examined attempts by the Scholastics to mathematize change before algebra was available [**4**], the large literature on students' difficulties with kinematics [**8**] and graphs [**6**], and a view that new technologies could be used to support learning that is more foundational than learning facility with traditional notations. Rather, we wished to build the ideas to which these notations conceptually refer, the ideas that they are "about." We asked how the key underlying ideas of rate of change, accumulation, the connections between variable rates and accumulation, and approximation all might be made sensible to as young and diverse a population as possible. This initially meant the middle school grades, but has more recently involved the elementary grades. Following the historical lead and recognizing that the language and metaphors of motion are used quite generally to describe change and variation [**2**], we focused (although not exclusively) on mathematizing linear motion, particularly by controlling motion simulations in familiar or fanciful situations: elevators, people walking or dancing, cars, duckies on a pond, boats in a river, space-vehicles, and so on [**10**].

Our starting criteria were to begin with students' intuitive experience with velocity, to minimize computational complexity, and yet to maintain sufficient variation to avoid the conceptual degeneracy of constant velocity [**12**]. These criteria led to extensive use of piecewise constant velocity functions. Furthermore, we wanted to support direct graphical manipulation of these velocity functions - after all, defining and manipulating piecewise constant functions algebraically is a very cumbersome process.

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[1] Funded by the NSF Applications of Advanced Technology Program, Grant #RED 9619102

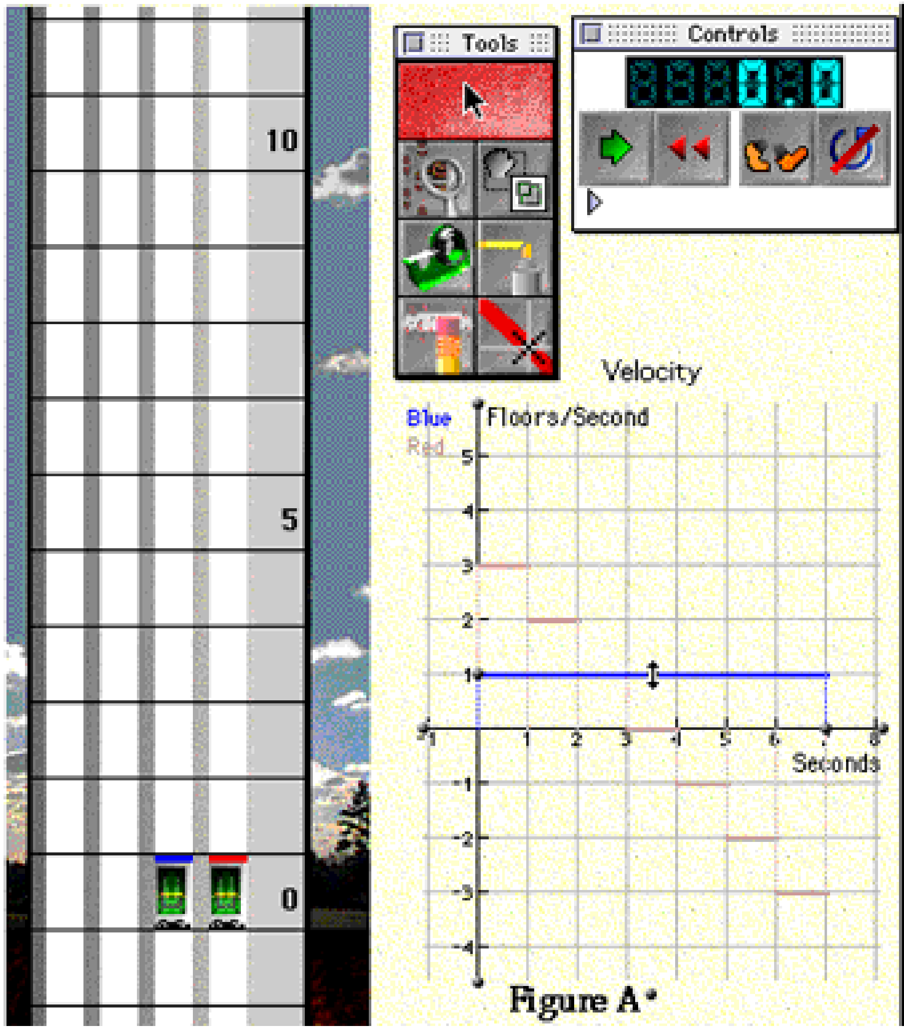

Figure A*

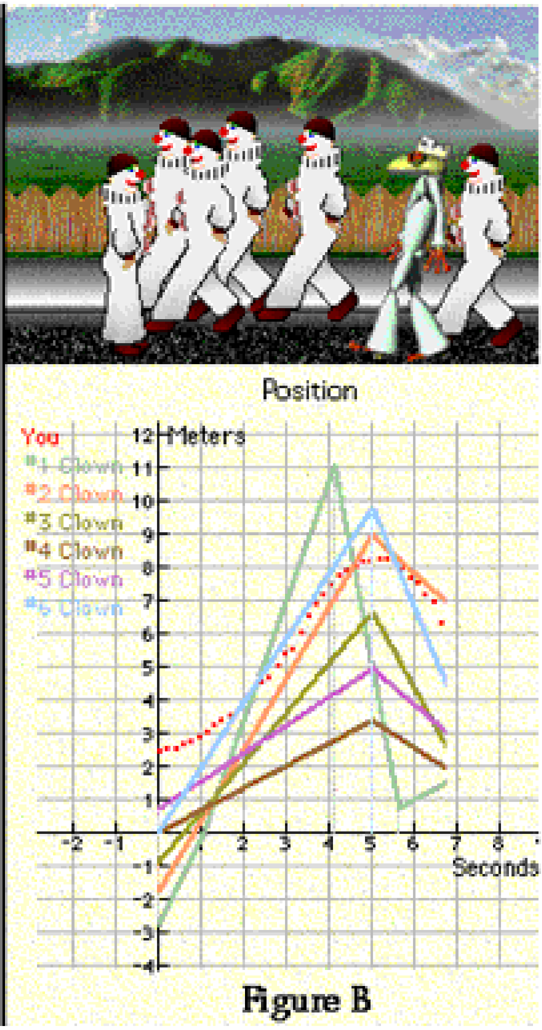

Figure B

These considerations lie behind the situation depicted in Figure A, where the graphs on the right side of the figure drive what we usually call "jerky elevators" on the left. Here the student is dragging vertically at the arrow a constant velocity segment (currently at height one floor/sec) in order to make the elevator named "Right" get to the same floor at the same time as elevator "Left," which has a discontinuously decreasing velocity (step) function. In this particular case, if "snap-to-grid" were turned on, forcing all values of time and velocity to be integers, the student could not succeed (9 floors in 7 seconds). Further, the Mean Value Theorem's continuity hypothesis is violated, of course, and its conclusion fails. If a linear velocity function had been used instead of the staircase, then the student would be building an instantiation of Merton's Theorem [**3**, p. 86].

Although it is difficult to build a case on such a narrow shard of curricular activity, this mathematically mundane problem-situation illustrates several aspects of rethinking subject matter as it is experienced and learned by students. First of all, three key underlying ideas - constant rate, mean-value, and area under a rate-graph - are directly and enactively addressed at a level sensible for upper elementary age students. Second, these ideas are approached graphically rather than algebraically, with a tight referential relationship to motion phenomena. Velocity, position, and acceleration graphs in this approach are not only linkable to *each other* (or to tables or equations), they also provide three different descriptions of readily viewable and controllable phenomena. That is to say, unlike much school activity based on the "Big Three" representations (numeric, graphic, and algebraic), *they represent something other than each other!* Their primary referential relation is to the phenomena. Third, accumulation, via simple arithmetic sums, is addressed before the subtle ideas of rate and slope.

We could also approximate the staircase by dragging a linear function into place - another interesting reversal of the usual direction, which is to approximate continuously varying functions by discretely varying ones. In our case the approximation-error can be directly computed and predicted by 5th graders, and can be tested by observing the final (or intermediate) positions of the elevators.

5. An Early Start: Building on Kinesthetic Experience. Young children (grades 2-5) first meet mean values in physical activities. Students have pre-quantified notions of their own "slow," "medium," and "fast" speeds that they enact on a marked section of classroom floor. A pair of students is to move along the marked line such that one (A) moves for 2 seconds at "slow" speed, then 2 seconds at "fast," and then 2 seconds again at "slow;" the other student (B) is to move at a constant "medium" such that she reaches the endpoint at exactly the same time as A. Perhaps after a practice run where B tries, at roughly constant speed, to cover the distance that A traveled in 6 seconds, they try to perform this task in parallel. Great struggles ensue as B tries not to be influenced by the fact that at first she is ahead of A and then is behind A. The observing students shout "Constant speed!" "No changing!" and so on, while B fights her own kinesthetic sense to slow down, to catch up, etc. She is learning, at a very fundamental physical and personal level, a version of constant rate, and perhaps less directly, a sense of average speed, that serves as a foundation for understanding constant functions and linearly increasing distance in their various more formal representations to come, including those encountered in simulations such as in Figure A. Indeed, children's physical experience is difficult for them to quantify but is kinesthetically rich, whereas simulations can be made quantitatively rich, although are kinesthetically vacuous. Hence we engage students in parallel activities on the computer, where they begin with piecewise constant velocity functions labeled only as "slow," "medium," and "fast" (one, two, and four floors/sec, respectively), and gradually move to more quantitative problems and methods, first graphical and numeric and, eventually, algebraic.

Another connection with physical motion is available through the use of motion sensors, which in microcomputer based labs have been used to import and then graph quantitative aspects of phenomena, approximate the data via curve-fitting techniques, etc. In SimCalc MathWorlds [**10**], it is possible to attach the motion-data to an object and replay it, or edit the motion, or, as was done in Figure B, create a series of motions that relate in some interesting way to the original motion.

Here the Frog-character, with the dashed position graph representing a student's actual imported motion, is leading a "Clown Parade" where the clowns were given position functions synthetically. Note the "class-clown" outlier who is marching to her own drummer. (Software graphics are in color, allowing color-coded graphs and actors). The next generation of this software will support uploading of functions from hand-held devices, so each student in the class can control their own character in the simulation - they can "be in" the parade!

6. Ongoing Investigations: Formalization. An important question is how to get from the informal understandings discussed above to a more formal calculus that supports the symbolic technique that *some* students need. We closely study student work in various activities and environments that both lays the base for, and then actually builds, the many ideas at the heart of C-KNOW. As such ideas are being solidly established, their formalization, which is the source of enormous power for those who have mastered it but a great difficulty for many students at all levels, becomes a much more tractable matter. Two formalization strategies are available. One involves beginning with an extended, graphical pre-algebraic mathematics of change experience in the earlier grades and then superimposing algebraic notation upon what is already understood graphically, representing the graphs and phenomena via the usual classes of functions. The second involves co-learning algebra and the ideas that it can represent and manipulate, including rule-based descriptions of motion and change. The second strategy, which perforce must begin in the early grades, introduces and reveals the computational power of the formalisms early and often - as was the case historically. Both strategies provide an experiential anchor for some of the basic functions, their derivatives and integrals. And both offer a major departure from remodeling C-INST for the privileged 10% to building new curricular structures to include ALL students.

The combination of factors that led to Calculus reform applies across the mathematics curriculum, as does the inadequacy of top-down, university-centric reforms. Just as previously successful rote-based learning and coping strategies prove inadequate for students as the mathematical challenges become more substantial, inherited curricular strategies and technologies prove inadequate in the face of our historic challenge to teach much more mathematics to many more people than ever before. And, to go a step farther, our reform strategies may prove inadequate as well. We now must reach beyond the confines of the university, and draw upon deep and detailed understandings of mathematical learning and thinking across many age levels and types of students. This is neither as easy nor as convenient as speculation about neat new topics, but without more foundational work our, and our students', real alternatives will remain limited.

References

1. J. Casti, *Would-be worlds: How simulation is changing the frontiers of science,* John Wiley & Sons, New York, 1996.
2. M. Claggett, *Nicole Oresme and the medieval geometry of qualities and motions, Chapters I & II,* University of Wisconsin Press, Madison, WI, 1968.
3. C. Edwards, *The historical development of the calculus*, Springer-Verlag, Inc., New York, NY, 1979.
4. J. Kaput, Democratizing access to calculus: New routes using old roots, in: *Mathematical thinking and problem solving,* A. Schoenfeld, Editor, Erlbaum, Hillsdale, NJ, 1994, 77-156.
5. M. Kline, Logic vs. pedagogy, *Amer. Math. Monthly* 77 (1970), 264-282.
6. G. Leinhardt, O. Zaslavsky and M. Stein, Functions, graphs, and graphing: Tasks, learning, and teaching, *Review of Educational Research* 60 (1990), 1-64.
7. S. MacLane, Calculus is a discipline, *College Math. J.* 15 (1984), 375.
8. L. McDermott, M. Rosenquist and E. v. Zee, Student difficulties in connecting graphs and physics: Examples from kinematics, *Ameri.J. Physics* 55 (1987), 503-513.
9. A. Ralston, Computer science, mathematics and the undergraduate curricula in both, *Amer. Math Monthly* 88 (1981), 472-485.
10. SimCalc MathWorlds, software and associated materials available at http://www.simcalc.umassd.edu
11. I. Stewart, Change, in: *On the Shoulders of Giants: New approaches to numeracy,* L. Steen, Editor, National Academy Press, Washington, DC, 1990, pp.183-219.
12. W. Stroup, Embodying a nominalist constructivism, Unpublished doctoral dissertation, Harvard Graduate School of Education, 1996.
13. U.S. Department of Education, *The Condition of Education 1996, Supplemental Table 29-6,* (On-line) National Center For Education Statistics, Washington, DC, 1996. Available at http://www.ed.gov/NCES/pubs/ce/c9629d06.html.

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